PAPER P-1113

ON SIMULATIONS OF THE STOCHASTIC, HOMOGENEOUS, LANCHESTER LINEAR-LAW ATTRITION PROCESSS

Alan F. Karr

September 1975

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INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER P-1113	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On Simulations of the Stochastic, Homogeneous, Lanchester Linear-Law Attrition Process		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Alan F. Karr		8. CONTRACT OR GRANT NUMBER(s) IDA Independent Research Program
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Defense Analyses 400 Army-Navy Drive Arlington, Virginia 22202		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE July 1975
		13. NUMBER OF PAGES 31
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document is unclassified and suitable for public release.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lanchester linear attrition process, stochastic attrition model, combat model, Monte Carlo combat simulation, force ratio

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Results of computerized Monte Carlo simulations of a stochastic, homogeneous Lanchester-linear attrition process are presented. Properties investigated include expected numbers of survivors as functions of time, iterated calculations of expected numbers of survivors, probabilities of forcing the opposing side below a certain fraction of initial strength before being so reduced and relations of the

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20. stochastic model to the deterministic Lanchester-linear model.

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PREFACE

In this paper, we describe the results of some computerized Monte Carlo simulations of a stochastic attrition process analogous to the Lanchester "linear-law" differential-equation model of combat. The problems studied include expected numbers and distributions of survivors as functions of time, the effects of iterative approximations to expected numbers of survivors, and the probability of each side's "winning" an engagement in which the termination rule is of the form of breakpoints (not necessarily equal) for each side. In the latter case, we also consider distributions of numbers of survivors.

Chapter I

THE MATHEMATICAL MODEL

The deterministic model of which the stochastic process studied here is an analogue was introduced by F. W. Lanchester (Reference [6]). After presenting his "square law" as a model of combat in which the numerically superior side is able to bring that superiority to bear on the opposition, Lanchester turned to the description of combat that occurs in the sense of one-on-one engagements, so that the numerically superior side has an advantage only in having more eligible combatants. Lanchester proposed the model

$$b'(t) = -c_1 b(t) r(t) \quad \text{and} \quad r'(t) = -c_2 b(t) r(t) , \qquad (1)$$

where $b(t)$ and $r(t)$ are the numbers of surviving Blue and Red combatants, respectively, t time units after the combat begins. Here c_1 and c_2 are positive constants.

We make the following observations for purposes of analysis of simulation results:

If

$$c_1 r(0) = c_2 b(0) ,$$

then

$$\lim_{t\to\infty} b(t) = \lim_{t\to\infty} r(t) = 0.$$

Otherwise, $\lim b(t) > 0$ and $\lim r(t) = 0$ or $\lim b(t) = 0$ and $\lim r(t) > 0$ according as $c_1 r(0) < c_2 b(0)$ or $c_1 r(0) > c_2 b(0)$. Thus, the force ratio

$$f = \frac{c_1 r(0)}{c_2 b(0)}$$

determines the victor when the engagement is continued until one side or the other is annihilated.

If

$$c_1 = c_2 ,$$

then

$$b(t) - r(t) = b(0) - r(0) \tag{2}$$

for all t. Equation (2) confirms the nature of the model as a description of combat occurring in the form of one-on-one duels and of the role of numerical superiority. Here the originally superior side is unable to increase its *absolute* superiority, which it is able to do in square-law combat; cf. Reference [5, p. 10].

The stochastic process we have simulated is the homogeneous linear-law process L1 of Reference [4]. For completeness, we list here the assumptions from which that process is derived, along with a description of the process as a regular Markov process.

Assumptions

(1) All combatants on each side are identical.

(2) The time required for a particular Blue combatant to detect a particular Red combatant is exponentially distributed with expectation $1/d_b$, Each Blue combatant detects different Red combatants independently.

(3) A Blue combatant attacks every Red combatant it detects; the conditional probability of kill, given detection and attack, is k_b. The attack occurs instantaneously, and contact is lost immediately thereafter. No attack can occur without a detection.

(4) Red combatants satisfy the same assumptions with parameters d_r and k_r.

(5) The detection and attack processes of all combatants are mutually independent.

One can construct a probability space $(\Omega,\underline{\underline{M}},P)$ and on it a stochastic process $((B_t,R_t))_{t\geq 0}$ with state space $E = \underset{\sim}{N} \times \underset{\sim}{N}$ (where $\underset{\sim}{N} = \{0,1,2,\cdots,\}$) such that the following characterization holds. The interpretation is that B_t is the (now random) number of Blue combatants surviving at time t and that R_t is the corresponding number of Red combatants.

THEOREM. Under Assumptions (1)-(5), the stochastic process $((B_t,R_t))_{t\geq 0}$ is a regular Markov process, with jump function λ given by

$$\lambda(i,j) = ij(d_b k_b + d_r k_r) ,$$

transition kernel P given by

$$P\big((i,j);(i-1,j)\big) = \frac{d_r k_r}{d_b k_b + d_r k_r}$$

$$P\big((i,j);(i,j-1)\big) = \frac{d_b k_b}{d_b k_b + d_r k_r} , \qquad (3)$$

and infinitesimal generator Q given by

$$Q\big((i,j);(i-1,j)\big) = ijd_r k_r$$

$$Q\big((i,j);(i,j)\big) = -ij(d_b k_b + d_r k_r)$$

$$Q\big((i,j);(i,j-1)\big) = ijd_b k_b . \qquad (4)$$

The assertion that this stochastic process is an appropriate analogue of the deterministic model (1) is based upon the resemblance of (1) and the infinitesimal generator Q given in Equation (4). (Reference [4] gives a detailed explanation of this analogy.)

The reader is referred to Reference [3] for further details, interpretations, and proofs concerning regular Markov processes. By a well-known characterization of regular Markov processes (Reference [1]), we have the following interpretation: When the stochastic attrition process $((B_t,R_t))_{t\geq 0}$ enters a state (i,j), it remains there an exponentially distributed sojourn

which is independent of all past history of the process and has expectation $1/\lambda(i,j)$. At the end of this sojourn, it jumps to a new state according to the distribution $P\big((i,j);\cdot\big)$ independently of all past history, including the length of the sojourn. The sequence $\big((\hat{B}_n,\hat{R}_n)\big)_{n\in\tilde{N}}$ of states visited forms a Markov chain (the *imbedded Markov chain*) with transition kernel P given by Equations (3). Except along the axes, this Markov chain is a homogeneous random walk--an observation that is useful in the analysis of some of our simulation results.

We denote by $P^{(i,j)}$ the probability law of the attrition process $\big((B_t,R_t)\big)$ subject to the initial conditions

$$B_0 = i \quad \text{and} \quad R_0 = j$$

and by $E^{(i,j)}$ the corresponding expectation operator.

The details of the simulation procedure are essentially those used in a previous simulation study of a stochastic square-law attrition process (we refer the reader to Reference [5] for details).

Chapter II

EXPECTED NUMBERS AND DISTRIBUTIONS OF SURVIVORS

We begin with a rather detailed attempt to ascertain the shapes of the functions

$$t \to E^{(i,j)}[B_t] \quad \text{and} \quad t \to E^{(i,j)}[R_t]$$

for various choices of initial force strengths (i,j) and fixed values of the parameters d_b, d_r, k_b, and k_r. In Table 1, we summarize the results obtained.

Table 1 contains three sets of data, corresponding to

$$B_0 = R_0 = 200 \ ;$$

$$B_0 = 50 \quad \text{and} \quad R_0 = 60 \ ;$$

$$B_0 = 50 \quad \text{and} \quad R_0 = 55 \ .$$

The most striking aspect of all these results is the great intensity of the early stages of the engagement. In Figure 1, we give graphically the progress of the battle. Since, for all t, we very nearly have $E[B_t] = E[R_t]$ (the two forces are initially exactly equal), we have given one graph for both sets of data. One can observe that each side is reduced to half its initial strength by $t = 0.1$, even though the mean time to detect a particular member of the opposition is $1/d_b = 1/d_r = 10$. Initial intensity is greater for linear-law combat than for square-law combat with appropriately analogous parameters.

Figure 2 represents the same data, but with the time axis logarithmic. The figure shows that the attrition begins very intensely, and then moderates as the engagement progresses.

Table 1. EXPECTED NUMBERS OF SURVIVORS

$[d_b = d_r = 0.1 \quad k_b = k_r = 0.5]$

t	$E^{(i,j)}[B_t]$	$E^{(i,j)}[R_t]$
0.000	200.00	200.00
0.100	99.20	100.64
0.200	67.36	65.94
0.300	49.48	52.64
0.400	40.34	41.20
0.500	34.22	33.52
0.600	29.86	30.30
0.000	200.00	200.00
0.010	181.40	181.86
0.020	166.96	165.98
0.030	153.44	153.22
0.040	143.12	142.66
0.050	133.02	133.62
0.060	124.66	125.80
0.070	117.74	118.00
0.080	111.18	110.64
0.090	105.34	106.44
0.100	100.10	100.10
0.000	200.00	200.00
0.001	197.96	198.28
0.002	195.76	196.28
0.003	194.58	194.56
0.004	192.48	192.34
0.005	190.50	190.26
0.006	188.40	189.02
0.007	187.42	187.02
0.008	184.96	186.10
0.009	183.04	183.86
0.010	182.58	181.52
0.000	50.00	60.00
1.000	10.60	19.86
2.000	4.96	14.72
3.000	3.18	13.04
4.000	2.44	11.62
0.000	50.00	60.00
0.100	38.56	48.18
0.200	30.02	41.52
0.300	25.18	35.84
0.400	22.24	30.14
0.500	18.48	28.10
0.000	50.00	60.00
0.010	48.32	58.58
0.020	47.08	57.18
0.030	45.98	56.04

t	$E^{(i,j)}[B_t]$	$E^{(i,j)}[R_t]$
0.040	44.78	55.22
0.050	43.58	53.02
0.060	42.54	51.72
0.070	41.20	51.74
0.080	39.98	50.20
0.090	39.20	49.42
0.100	38.76	47.54
0.000	50.00	60.00
0.001	49.84	59.80
0.002	49.76	59.68
0.003	49.64	59.62
0.004	49.40	59.44
0.005	49.34	59.36
0.006	49.18	59.16
0.007	48.82	58.88
0.008	48.80	58.74
0.009	48.84	58.76
0.010	48.72	58.58
0.000	50.00	55.00
1.000	13.04	16.86
2.000	6.28	12.92
3.000	4.70	10.14
4.000	3.54	8.36
5.000	3.16	8.08
0.000	50.00	55.00
0.100	38.86	44.26
0.200	31.76	36.34
0.300	26.52	32.14
0.400	23.02	28.24
0.500	20.68	25.32
0.600	17.36	23.92
0.700	16.58	22.04
0.800	14.94	19.58
0.900	13.78	18.22
0.000	50.00	55.00
0.010	48.72	54.02
0.020	47.28	52.26
0.030	46.96	51.42
0.040	44.90	50.14
0.050	44.24	49.16
0.060	42.76	48.10
0.070	41.86	46.66
0.080	40.40	46.54
0.090	40.20	44.86
0.100	39.94	43.26

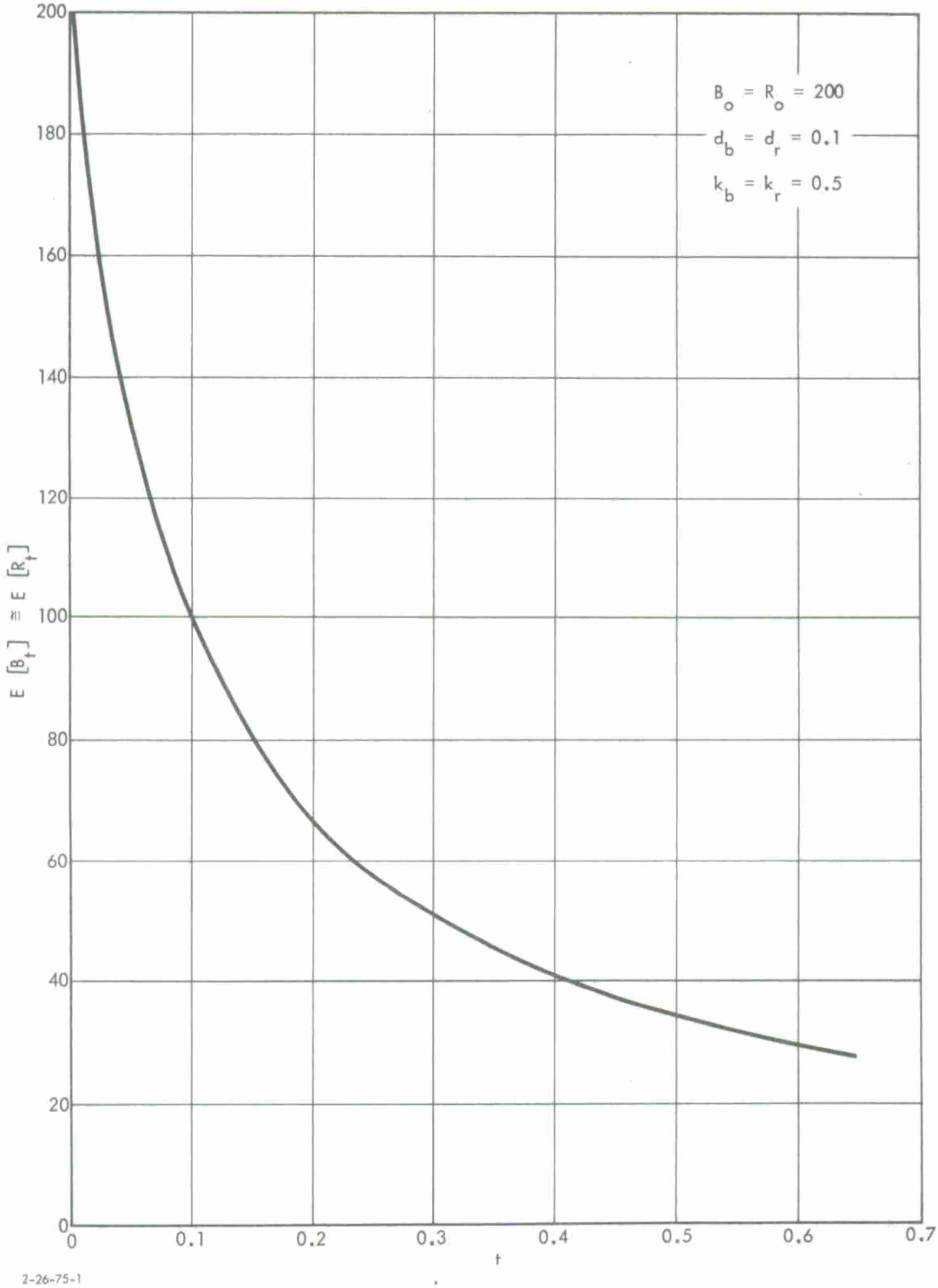

Figure 1. EXPECTED NUMBERS OF SURVIVORS (DETECTION RATE = 0.1)

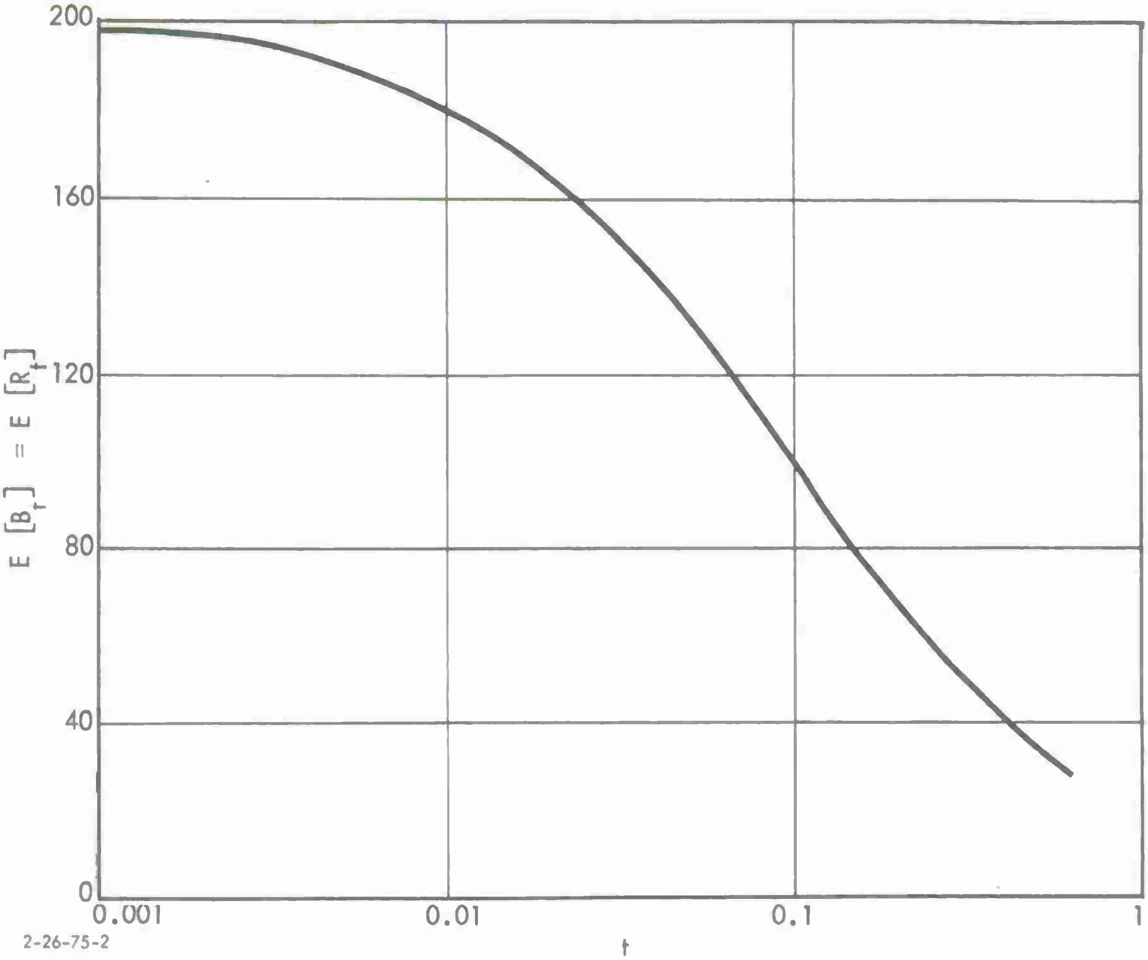

Figure 2. EXPECTED NUMBERS OF SURVIVORS (Logarithmic Scale)

Neither graph indicates a simple form for the function $t \to E[B_t] = E[R_t]$. It is not, for example, a linear function of log t. One reason for undertaking these investigations that use Monte Carlo simulations was our inability to obtain a tractable analytical expression for these functions (cf. Reference [5] for some pertinent comments concerning the difficulties involved). The simulations have given us a qualitatively clear picture (intense attrition at the start, becoming less intense as time elapses), but the analytical problems remain and are worthy of further research effort.

The other two sets of data in Table 1 exhibit substantially the same qualitative behavior and in this respect will not be

discussed further. A more interesting aspect of these data is that they seem to exhibit the stochastic analogue (in the sense of expectations) of the property, expressed in Equation (2), of preservation of absolute numerical advantage. Namely, in both cases, we nearly have

$$E^{(i,j)}[B_t - R_t] = B_0 - R_0 = i - j$$

for all t. (See Figure 3 for a graphical presentation of the third set of data in Table 1.) We are led, therefore, to the following:

CONJECTURE. If $d_b k_b = d_r k_r$, then for all initial conditions (i,j) and all $t \geq 0$ we have

$$E^{(i,j)}[B_t - R_t] = i - j \ . \tag{5}$$

To see why this conjecture should be true, consider the imbedded Markov chain $\big((\hat{B}_n,\hat{R}_n)\big)$. We warn the reader that $(\hat{B}_n,\hat{R}_n) \neq (B_n,R_n)$; indeed, $(\hat{B}_n,\hat{R}_n) = (B_{T_n},R_{T_n})$, where T_n is the random time of the n^{th} change of state of the continuous time process $\big((B_t,R_t)\big)$. In this case, by Equations (3),

$$P\big((i,j);(i-1,j)\big) = P\big((i,j)=(i,j-1)\big) = 1/2$$

shows that we have, with respect to $P^{(i,j)}$, the representation

$$(\hat{B}_n,\hat{R}_n) = (i,j) + \sum_1^n (X_k,Y_k), \qquad n \leq \min\{i,j\} \ ,$$

where (X_1,Y_1) and $(X_2,Y_2),\cdots$ are independent and identically distributed random variables with

$$P^{(i,j)}\{(X_k,Y_k)=(-1,0)\} = P^{(i,j)}\{(X_k,Y_k)=(0,-1)\} = 1/2 \ .$$

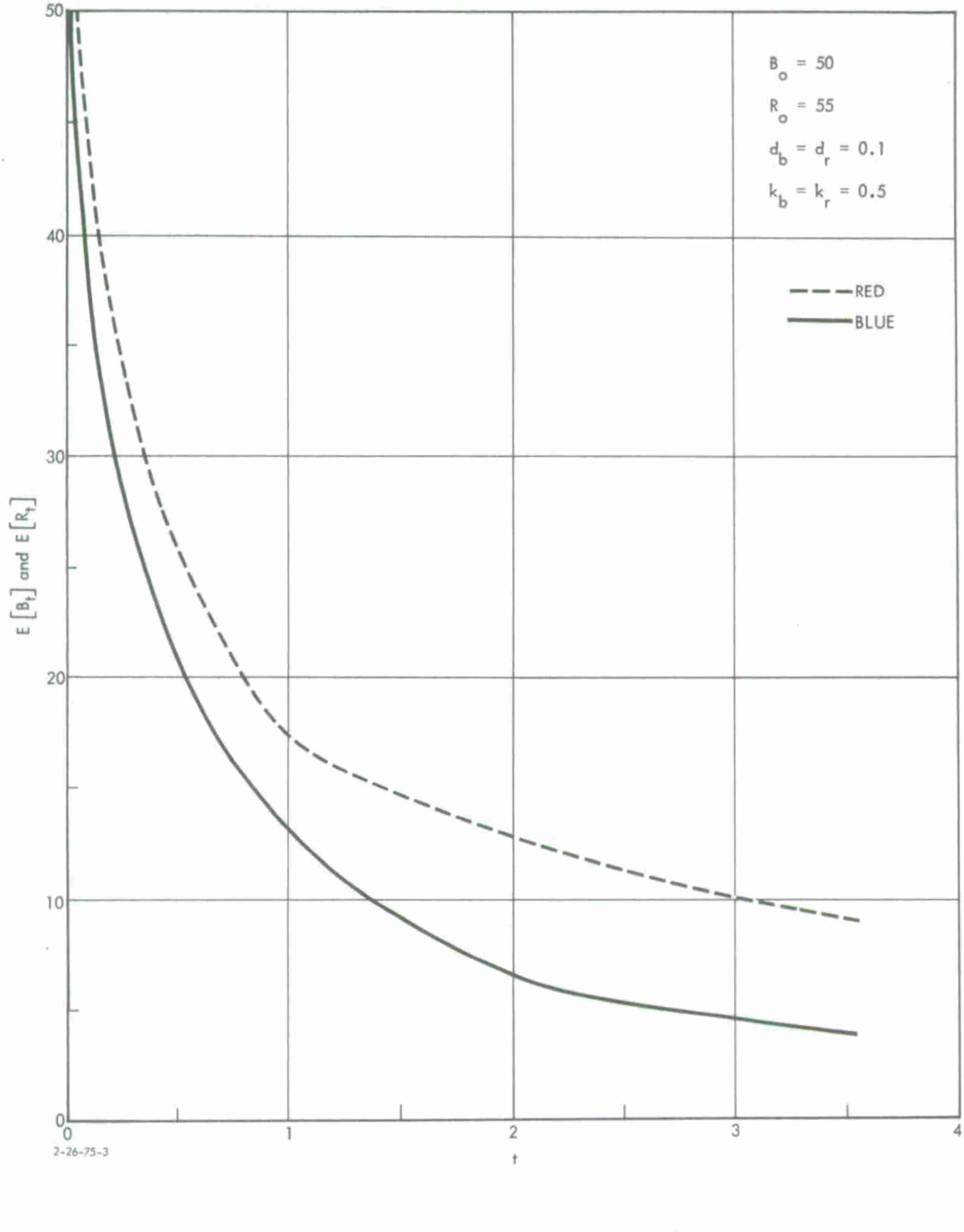

Figure 3. EXPECTED NUMBERS OF SURVIVORS (Unequal Forces)

Thus, it is clear that

$$E^{(i,j)}[\hat{B}_n - \hat{R}_n] = i - j \tag{6}$$

for $n \leq \min\{i,j\}$. For other n, the argument becomes more complicated, but Equation (6) remains true. The only step remaining is to pass from Equation (6) to Equation (5), which should not be too difficult.

In Figures 4, 5, and 6, we give the probability distribution of the state of the process at times 0.3125, 1.25, and 5 for the parameter values

$$B_0 = R_0 = 100 \ ;$$
$$d_b = d_r = 0.1 \ ;$$
$$k_b = k_r = 0.5 \ .$$

In all figures, the digit appearing at a particular location (x,y) is the number of realizations of the simulation for which $(B_t,R_t) = (x,y)$.

Two observations concerning these results seem warranted:

(1) Even for small times (i.e., $t = 0.3125$, in Figure 4), there is substantial variance in the vector random (B_t,R_t). There is also rather evident symmetry in its distribution. As time increases, the variance, as one would certainly predict, increases also.

(2) For $t = 0.3125$ (Figure 4) and $t = 1.25$ (Figure 5), there is a negative correlation between B_t and R_t, which is (visually, at least) remarkably close to -1. One might have so expected on intuitive grounds; nonetheless, it is illuminating to have a confirmation. For $t = 5$ (Figure 6), the negative correlation is present but not -1, because of the possibility (indeed, frequency) of annihilation. To us, this negative correlation is a rather strong argument against the use of deterministic models, even though (as seen in the next chapter) the symmetry of these distributions allows iterative calculations of *expected* numbers of survivors. This is because (with initially equal forces) in a deterministic model attritions to the two sides are always equal; that is, have correlation equal to 1.

Clearly negatively correlated attritions are more physically plausible.

The reader may draw his own further conclusions from these data.

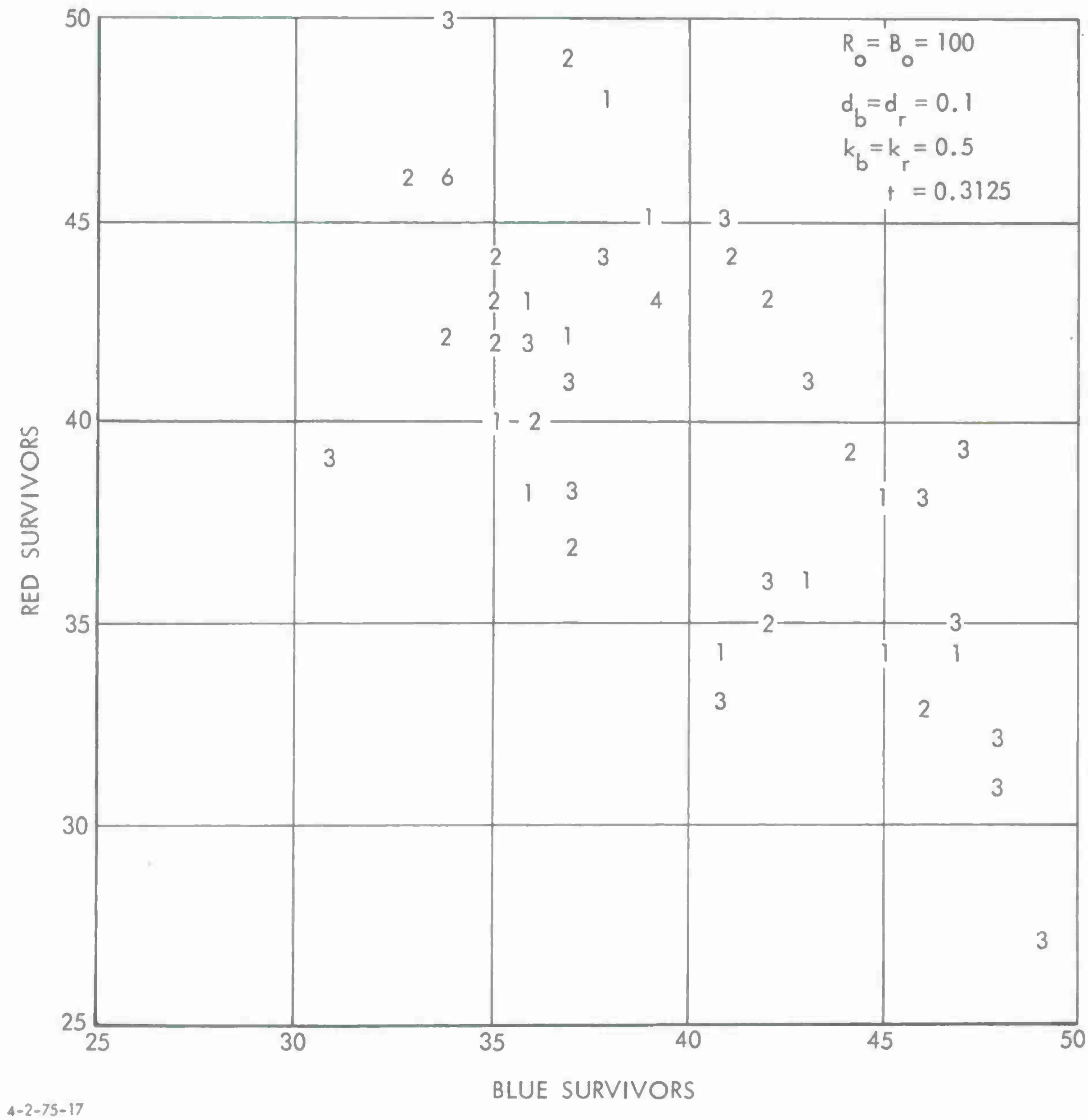

Figure 4. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Time = 0.3125)

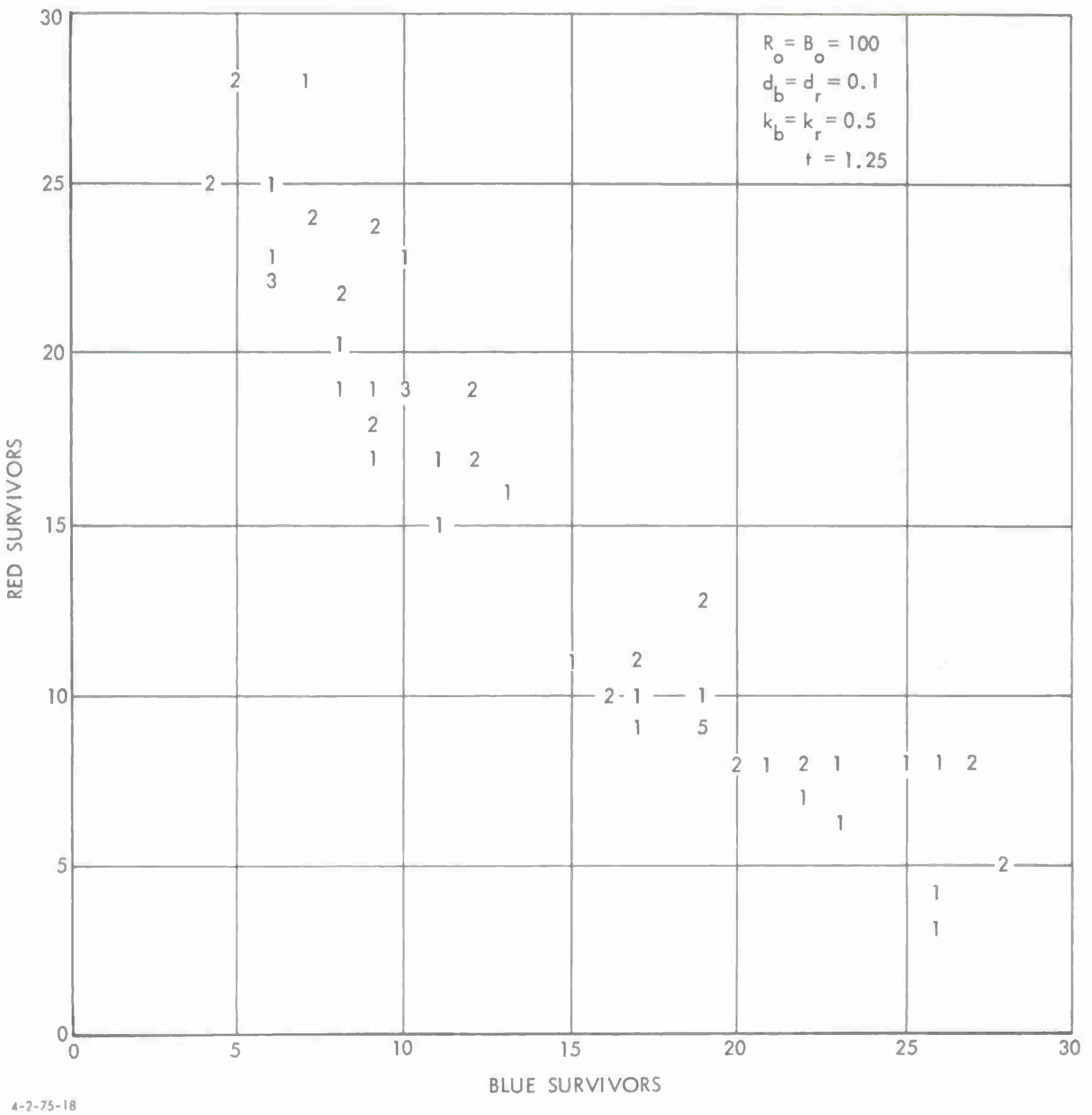

Figure 5. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Time = 1.25)

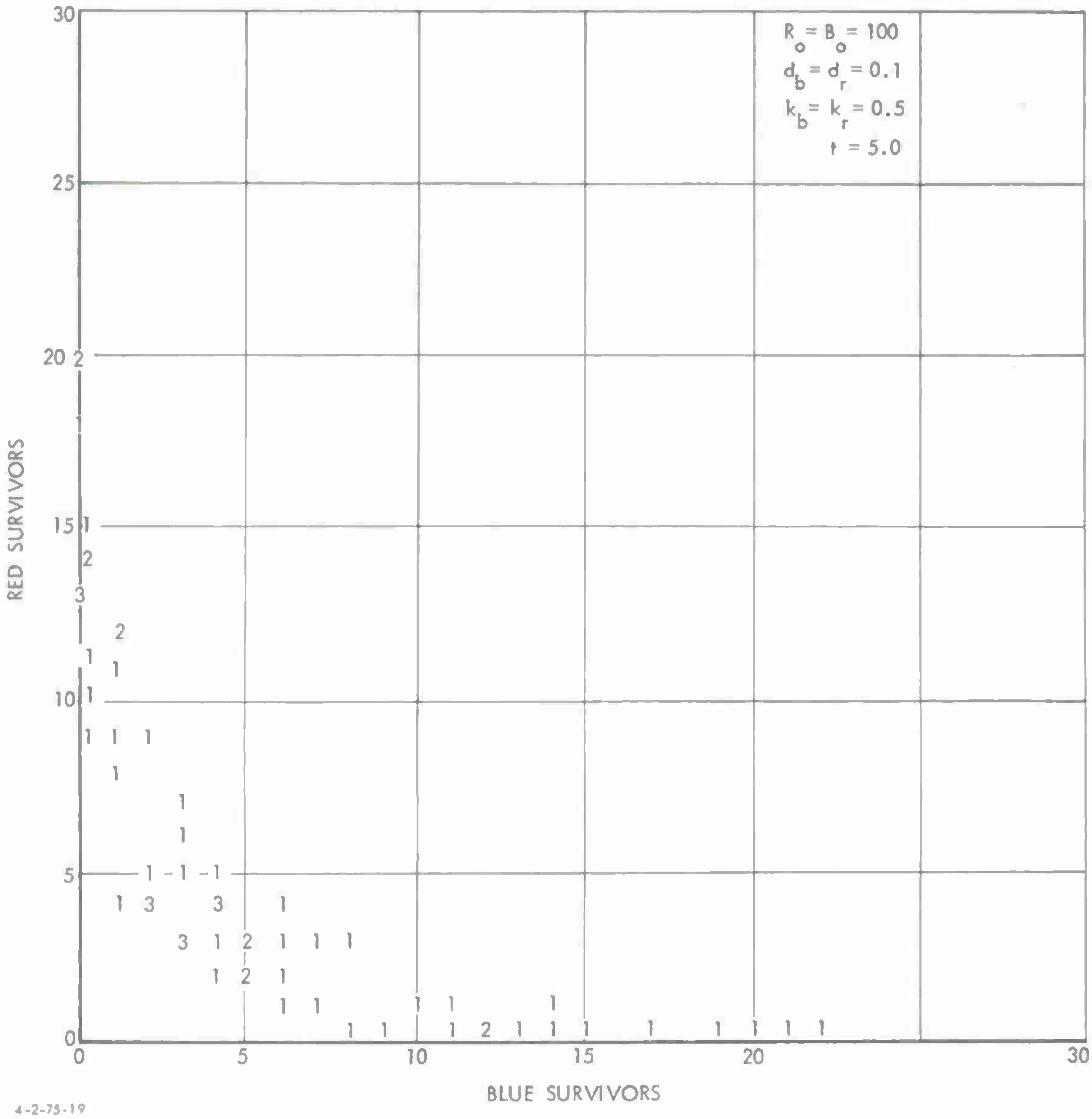

Figure 6. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Time = 5.0)

Chapter III

ITERATED CALCULATIONS

As discussed in more detail in Reference [5], it is of some interest in the context of the potential applicability of such stochastic attrition models (through the use of expectations) to iterative and deterministic combat simulations (e.g., IDAGAM I--Reference [1]) to consider the error in the approximation

$$E^{(i,j)}[B_{t_1+t_2}] = E^{(b,r)}[B_{t_2}] \,, \tag{7}$$

where

$$\begin{aligned}(b,r) &= E^{(i,j)}[(B_{t_1},R_{t_1})] \\ &= \left(E^{(i,j)}[B_{t_1}],\ E^{(i,j)}[R_{t_1}]\right) .\end{aligned}$$

In order that the question even make sense, one must extend the state space of the stochastic attrition process to be $[0,\infty) \times [0,\infty)$. This extension represents no real problem (details are the same as described in Reference [5] for the square-law process).

The results of these investigations, for several choices of parameter values and initial forces, appear in Table 2. The five sets of data exhibit the same qualitative features, the three most salient of which are the following:

(1) In absolute terms, the error committed by using the iterated calculation to approximate the correct calculation is small--never more than about 3. In relative terms, the error may be substantial

Table 2. ITERATED CALCULATIONS OF NUMBERS OF SURVIVORS

$B_0 = R_0 = 100;\ d_b = d_r = 0.1;\ k_b = k_r = 0.5$								
	Iteration Interval							
	0.5		1.0		2.0		4.0	
Time	Blue	Red	Blue	Red	Blue	Red	Blue	Red
0.0	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.5	28.08	28.88	--	--	--	--	--	--
1.0	16.62	16.82	17.54	17.32	--	--	--	--
1.5	11.70	11.62	--	--	--	--	--	--
2.0	9.42	8.80	9.86	9.02	10.66	11.10	--	--
2.5	7.78	7.10	--	--	--	--	--	--
3.0	6.62	5.96	7.02	5.82	--	--	--	--
3.5	5.90	5.22	--	--	--	--	--	--
4.0	5.10	4.84	5.36	4.58	5.14	5.52	7.82	6.48
$B_0 = 100;\ R_0 = 125;\ d_b = d_r = 0.1;\ k_b = k_r = 0.5$								
0	--	--	100.00	125.00	100.00	125.00	100.00	125.00
1	--	--	8.06	33.96	--	--	--	--
2	--	--	2.14	27.20	31.60	26.10	--	--
3	--	--	0.84	25.56	--	--	--	--
4	--	--	0.18	25.06	0.26	23.18	0.68	26.26
$B_0 = 100;\ R_0 = 125;\ d_b = d_r = 0.1;\ k_b = 0.5;\ k_r = 0.4$								
0	--	--	100.00	125.00	100.00	125.00	100.00	125.00
1	--	--	16.42	21.94	--	--	--	--
2	--	--	8.60	12.18	10.50	13.10	--	--
3	--	--	5.50	9.18	--	--	--	--
4	--	--	4.10	6.76	5.18	6.28	6.06	10.70
$B_0 = 100;\ R_0 = 150;\ d_b = d_r = 0.1;\ k_b = 0.5;\ k_r = 0.33$								
0	--	--	100.00	150.00	100.00	150.00	100.00	150.00
1	--	--	16.48	27.42	--	--	--	--
2	--	--	8.24	16.08	9.62	15.92	--	--
3	--	--	5.34	11.50	--	--	--	--
4	--	--	3.82	9.10	4.42	8.50	6.50	10.82
$B_0 = 200;\ R_0 = 250;\ d_b = d_r = 0.1;\ k_b = 0.5;\ k_r = 0.4$								
0	--	--	200.00	250.00	200.00	250.00	200.00	250.00
1	--	--	17.67	27.78	--	--	--	--
2	--	--	8.20	15.42	12.66	14.04	--	--
3	--	--	4.88	11.26	--	--	--	--
4	--	--	3.16	9.24	6.30	6.12	10.58	11.06

compared to surviving forces, but is never large relative to initial numbers of forces.

(2) The iterated calculation always underestimates the value obtained in the proper calculation, and the amount of underestimation increases with the number of iterations.

(3) The amount of underestimation by iterative calculations increases with time. Hence, for times at which attrition is small (compared to initial forces), the approximation is probably accurate enough for most applications. A difficulty, though, is that only over very short time periods is the relative attrition fairly low, because of the great initial intensity of linear law combat.

We have no theoretical explanation of the underestimates arising through iterative calculations.

Chapter IV

BREAKPOINTS AND FORCE EQUALITY

Here we consider the following situation: Suppose that for Blue and Red we have defined breakpoints q_b and q_r (each between zero and one) and that the combat terminates whenever $B_t \leq q_b B_0$ or $R_t \leq q_r R_0$. Further, we declare Red to be the winner if $B_t \leq q_b B_0$ occurs first. For various choices of force parameters and breakpoints, we are interested, then, in computing the probability of Red's winning.

In analytical terms, we seek to compute

$$P^{(i,j)}\{U < V\} ,$$

where U and V are the stopping times defined by

$$U = \inf\{t: B_t \leq q_b B_0\} \quad \text{and} \quad V = \inf\{t: R_t \leq q_r R_0\} .$$

This probability, however, can clearly be computed by using the imbedded Markov chain $((\hat{B}_n, \hat{R}_n))_{n \geq 0}$, in terms of which it is expressed as

$$P^{(i,j)}\{S < T\} ,$$

where

$$S = \inf\{n: \hat{B}_n \leq q_b \hat{B}_0\} \quad \text{and} \quad T = \inf\{n: \hat{R}_n \leq q_r \hat{R}_0\} .$$

Because of the particularly simple structure of the imbedded Markov chain, one should be able to compute these probabilities explicitly--a problem on which we plan future research.

We have obtained simulation results for cases of equal and unequal breakpoints, in which we fix force numbers and compute the probability of Red's winning as a function of the *Lanchester-*

linear force ratio

$$f = \frac{d_r k_r R_0}{d_b k_b B_0}, \tag{8}$$

which is varied by changing the value of the Red detection time $1/d_r$. The Lanchester-linear force ratio is, of course, the same quantity usually called just the "force ratio." We distinguish it because the appropriate force ratio depends on the stochastic model under consideration (for details, cf. Reference [5]).

The first breakpoints considered are $q_b = q_r = 0.5$ (the results are given in Table 3 below; some of these also appear in Figure 7).

The main conclusions to be drawn from these results are the following:

(1) For $i = j$, as i increases the graph of the function $f \to P^{(i,j)}\{\text{Red wins}\} \equiv p_i(f)$ becomes different from 0 or 1 over a decreasingly small set that contains the point 1 (at which we clearly have each side equally likely to win). Indeed, it is easy to show that

$$\lim_{i\to\infty} p_i(f) = \begin{cases} 0, & f < 1/2 \\ 1/2, & f = 1/2 \\ 1, & f > 1/2 \,. \end{cases}$$

The graph is also, consequently, increasingly steep over the set where it is neither zero nor one.

(2) The force ratio required to achieve a substantial (say 0.9) probability of victory is much less than that postulated by the "conventional wisdom," which holds that a force ratio of 2 or 3 is required for victory. In all the cases shown in Figure 7, a force ratio of 1.4 gives a 0.9 probability of victory. The larger the number of combatants (on both sides) the smaller the force ratio required for 0.9 probability of victory. These conclusions are of significant practical import.

In Table 4 and Figure 8, we present analogous results for the asymmetric breakpoints

Table 3. PROBABILITY OF WIN AS A FUNCTION OF FORCE RATIO (Equal Breakpoints)

$[q_b = q_r = 0.5]$

f	P(Red Wins)	f	P(Red Wins)	f	P(Red Wins)
$B_0 = R_0 = 50$		$B_0 = R_0 = 75$		$B_0 = 50, R_0 = 75$	
0.5	0.01	0.5	0.01	0.240	0.00
0.6	0.02	0.6	0.01	0.489	0.00
0.7	0.11	0.7	0.07	0.738	0.15
0.8	0.18	0.8	0.19	0.987	0.48
0.9	0.32	0.9	0.35	1.236	0.81
1.0	0.47	1.0	0.50	1.485	0.97
1.1	0.65	1.1	0.71	1.734	0.99
1.2	0.72	1.2	0.80	1.983	1.00
1.3	0.85	1.3	0.86	--	--
1.4	0.84	1.4	0.92	--	--
$B_0 = 50, R_0 = 100$		$B_0 = R_0 = 200$		$B_0 = R_0 = 500$	
0.4	0.00	0.75	0.01	0.75	0.00
0.5	0.00	0.80	0.05	0.80	0.00
0.6	0.03	0.85	0.16	0.85	0.04
0.7	0.09	0.90	0.22	0.90	0.16
0.8	0.23	0.95	0.38	0.95	0.35
0.9	0.36	1.00	0.47	1.00	0.54
1.0	0.54	1.05	0.58	1.05	0.71
1.1	0.69	1.10	0.76	1.10	0.81
1.2	0.76	1.15	0.81	1.15	0.93
1.3	0.83	1.20	0.92	1.20	0.96
1.4	0.92	1.25	0.93	1.25	0.99
1.5	0.94	--	--	--	--
1.6	0.97	--	--	--	--
1.7	0.99	--	--	--	--

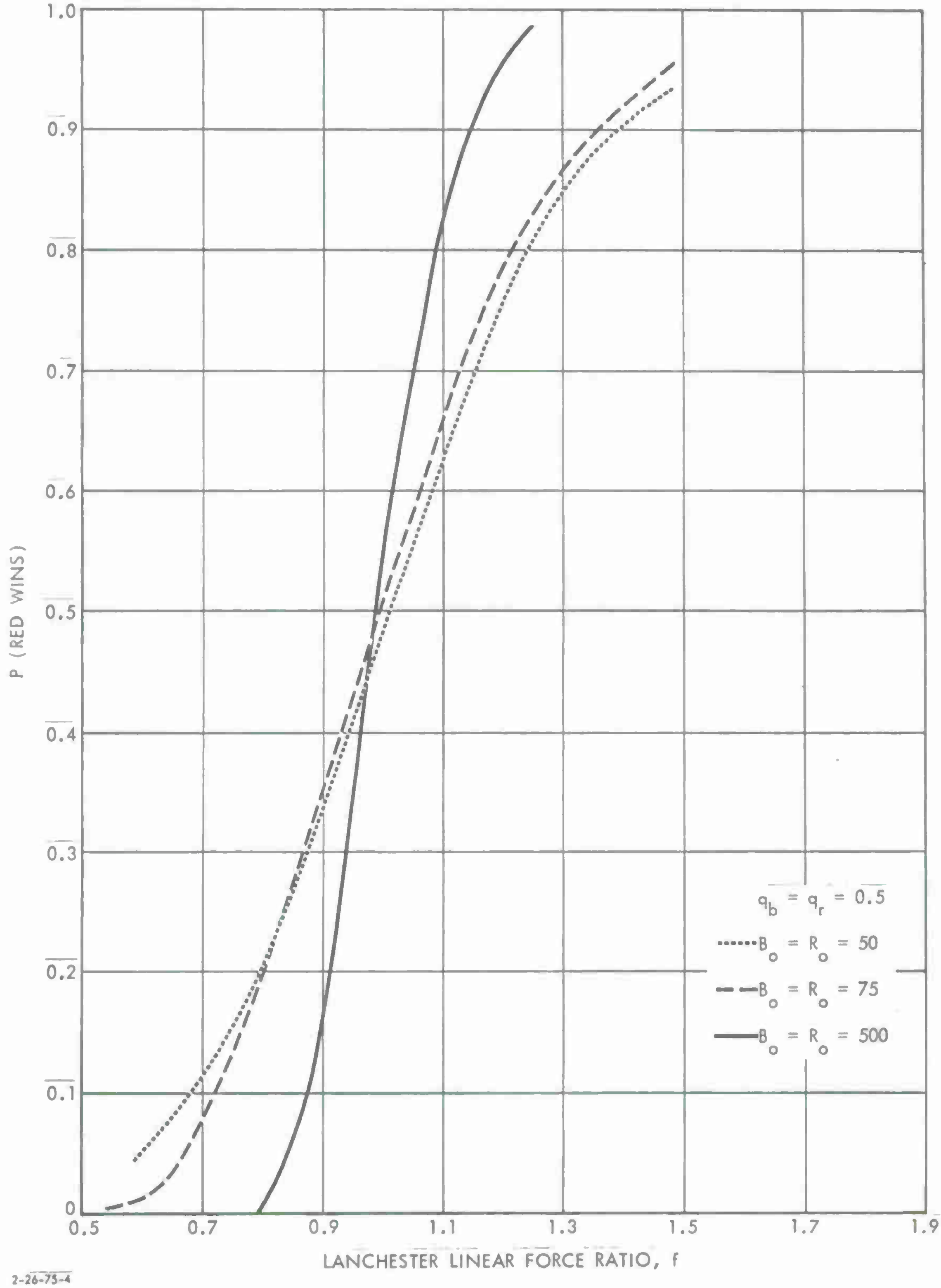

Figure 7. GRAPHICAL PRESENTATION OF TABLE 3

Table 4. PROBABILITY OF WIN AS A FUNCTION OF FORCE RATIO (Unequal Breakpoints)
[q_b = 0.67; q_r = 0.79]

f	P(Red Wins)	f	P(Red Wins)
$B_0 = R_0 = 50$		$B_0 = R_0 = 100$	
1.00	0.16	1.00	0.08
1.25	0.24	1.25	0.20
1.50	0.47	1.50	0.45
1.75	0.60	1.75	0.68
2.00	0.73	2.00	0.73
--	--	2.25	0.88
--	--	2.50	0.94
$B_0 = 50$, $R_0 = 75$		$B_0 = R_0 = 200$	
0.75	0.00	1.00	0.00
0.90	0.04	1.20	0.10
1.05	0.11	1.40	0.29
1.20	0.22	1.45	0.29
1.35	0.24	1.50	0.40
1.50	0.42	1.55	0.45
1.65	0.58	1.60	0.56
1.80	0.67	1.65	0.58
1.95	0.76	1.70	0.70
2.10	0.71	1.75	0.71
--	--	1.80	0.80
--	--	2.00	0.86
--	--	2.20	0.94
--	--	2.40	0.96
--	--	2.60	0.99
--	--	2.80	1.00

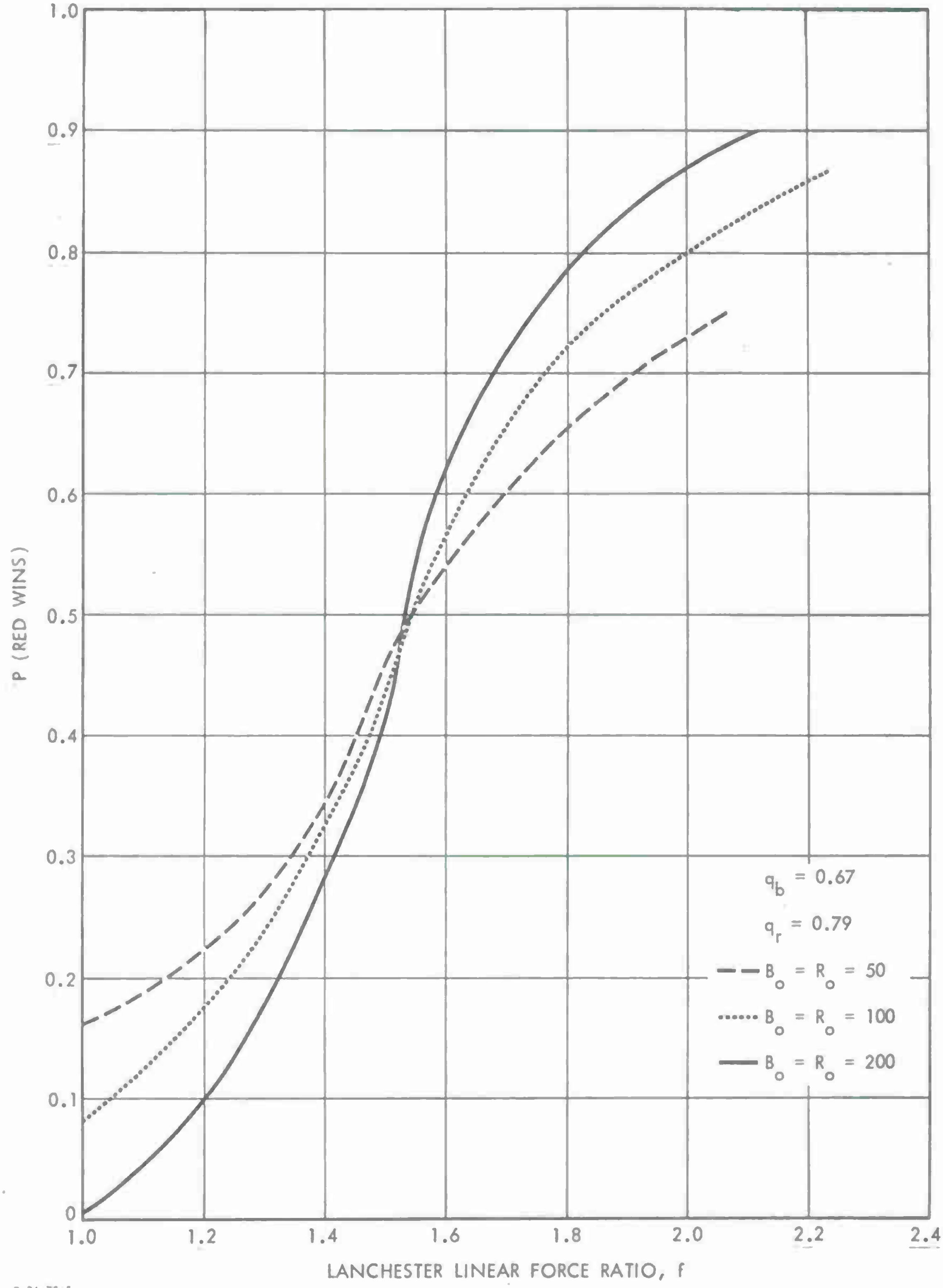

Figure 8. GRAPHICAL PRESENTATION OF TABLE 4

$$q_b = 0.67$$

$$q_r = 0.79 \ .$$

These figures are taken from the ATLAS model, where Red is the attacking side and Blue is the defending side.

The results are qualitatively the same as for the case of equal breakpoints; the reader may form and interpret his own conclusions. Of interest is the problem of solving the equation

$$p_i(f) = 1/2 \ , \qquad i > 0 \ .$$

From the data, it appears that the solution is possibly independent of i and equal approximately to 1.57. Noting that

$$\frac{1 - 0.67}{1 - 0.79} = 1.57143 \ ,$$

we are led to formulate the following assertion.

CONJECTURE. For all values q_b and q_r of the breakpoints and all values of the force effectiveness parameters,

$$p_i\left(\frac{1 - q_b}{1 - q_r}\right) = 1/2$$

for all i.

Hopefully, a proof will be devised soon.

In Figures 9, 10, 11, and 12, we give various simulated probability distributions for numbers of survivors in these situations. Figure 9 gives the distribution of the number of survivors on the winning side, for $R_0 = B_0 = 100$, f = 1, and $q_n = q_r = 0.5$. We note that the narrowest victories are the most likely. The expected surviving strength is approximately 55 percent. Figure 10 presents analogous data, but with f = 1.25. In this case, the defender (the weaker side) is likely to win only by a narrow margin, while the attacker's

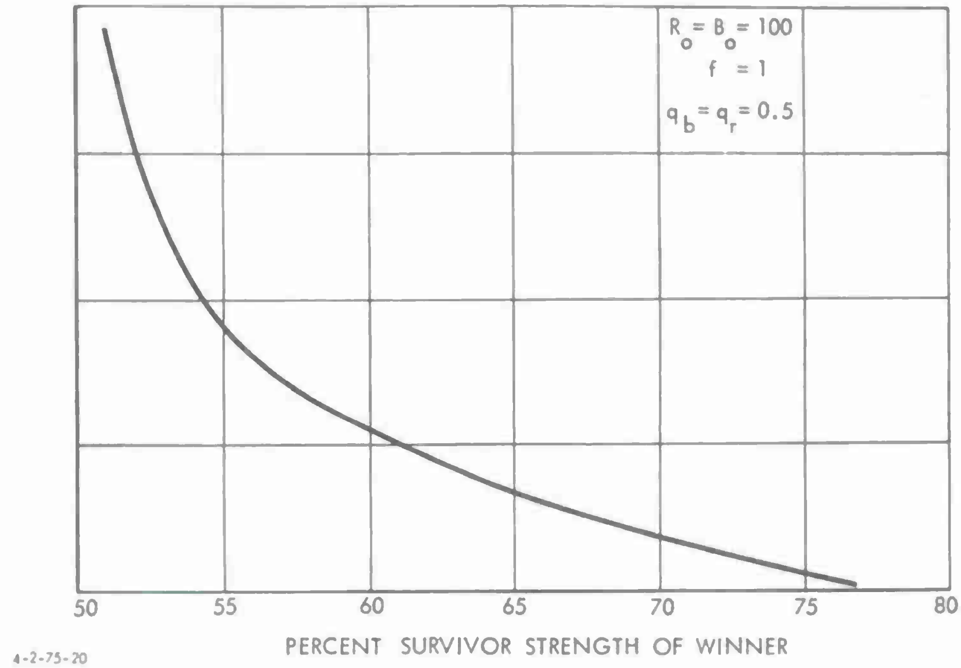

Figure 9. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Equal Forces)

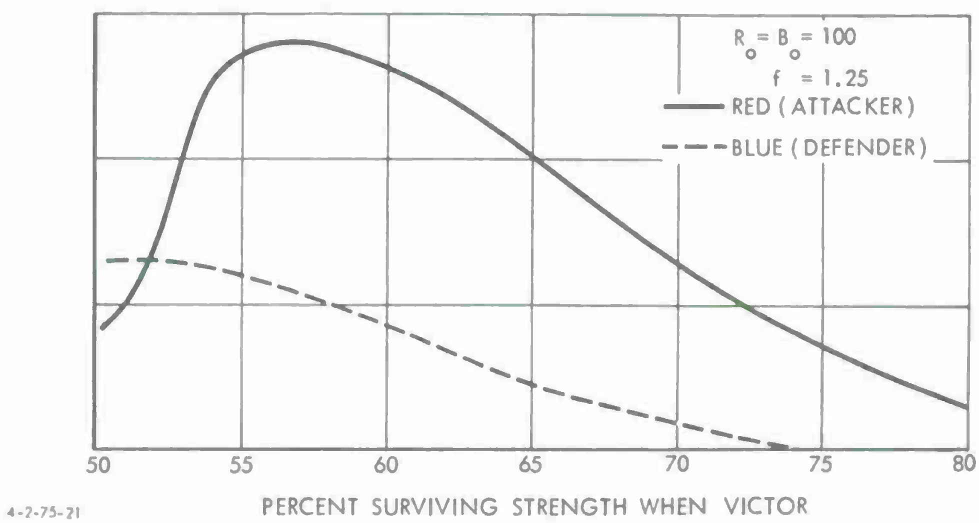

Figure 10. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Force Ratio = 1.25)

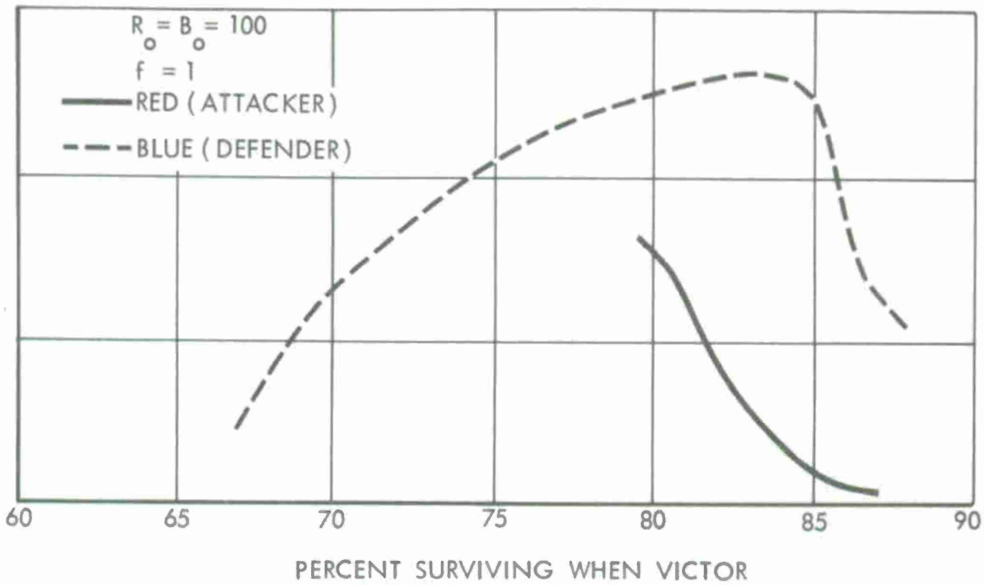

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Figure 11. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Unequal Breakpoints and Force Ratio = 1)

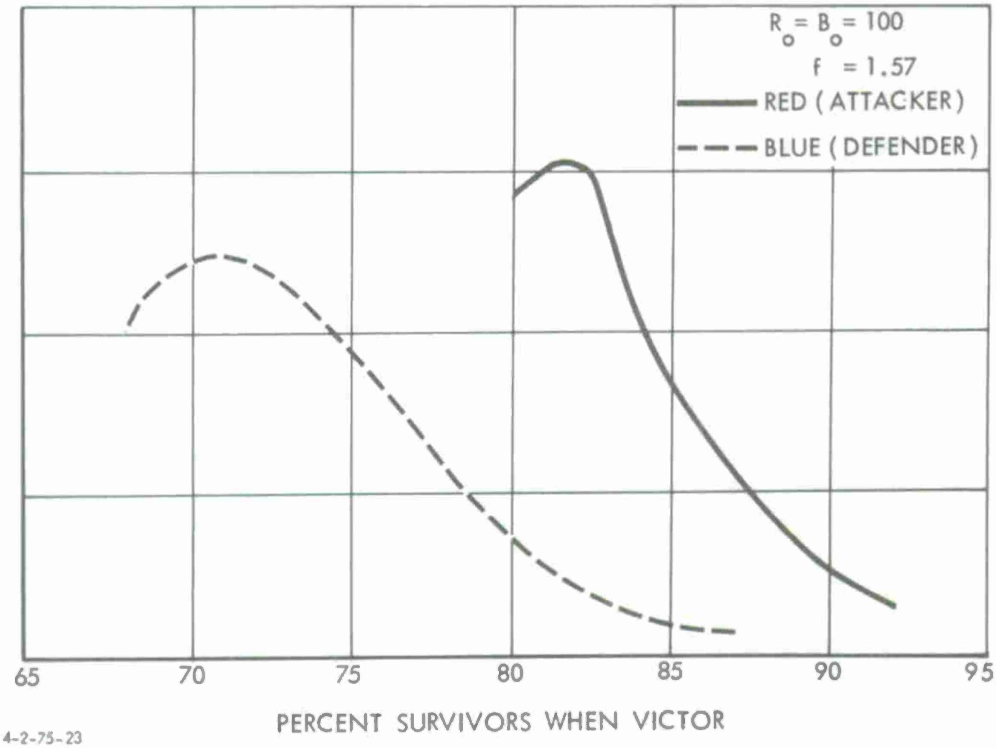

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Figure 12. PROBABILITY DISTRIBUTION OF NUMBERS OF SURVIVORS (Unequal Breakpoints and Force Ratio = 1.57)

most likely surviving strength is on the order of 55 percent, with expected surviving strength of approximately 60 percent.

The data in Figures 11 and 12 are based on the ATLAS breakpoints and on force ratios of 1 and 1.57 (approximate force equality, in the sense that each side is equally likely to win), respectively. The reader can form his own interpretations and draw his own conclusions. We remark again that overwhelming victories are not likely. In comparison with data concerning the stochastic square-law attrition process presented in Reference [5], the margins of victory are uniformly less in the linear-law case. These margins are expected, in the sense that our intuition concerning the two processes is that, in the square-law process, the numerically superior side can bring its full superiority to bear against the opposition but that it cannot do so in the linear-law process.

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